3. Vekua, N. P., Systems of Singular Integral Equations and Some Boundary Value Problems. 2d Ed., Moscow, "NAUKA" 1970.
4. Gakhov, F. D., Riemann boundary value problem for systems of n pairs of functions. Usp. Matem. Nauk, Vol. 7, No. 4, 1952.
5. Ufliand, Ia. S., Integral Transforms in Elasticity Theory. 2d Ed., Leningrad, "NAUKA", 1968.
6. Noble, B., Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations. Pergamon Press, 1959.
7. Lavrent'ev, M. A. and Shabat, B. V., Methods of Complex Variable Function Theory. 3d Ed. Moscow, "NAUKA", 1965.
8. Srivastav R. P. and Prem Narain, Certain two-dimensional problems of stress distribution in wedge-shaped elastic solids under discontinuous load. Proc. Cambr. Phil. Soc., Vol. 61, No. 4, 1965.
9. Bantsuri, R. D., Solution of the first fundamental problem of elasticity theory for a wedge having a finite slit. Dokl. Akad. Nauk SSSR, Vol. 167, No. 6, 1966.
10. Smetanin, B. I., Some problems of slits in an elastic wedge and layer. Inzh. Zh., Mekh. Tverd. Tela, No. 2, 1968.
11. Smetanin, B. I., On a mixed problem of elasticity theory for a wedge. PMM, Vol. 32, No. 4, 1968.

Translated by M. D. F.

ALGORITHM OF THE METHOD OF CHARACTERISTICS FOR THE ANALYSIS OF NONLINEAR ONE-DIMENSIONAL WAVE PROCESSES OF CONICAL AND CYLINDRICAL SHELL DEFORMATION

(PMM Vol. 35, No. 4, 1971, pp. 690-700)
A. Ia. LAKHE and U. K. NIGUL
(Tallin)
(Received March 2, 1971)

An algorithm is proposed for realizing the method of characteristics for the analysis of one-dimensional wave processes excited by the edge effect and described by a quasilinear system of differential equations having several pairs of families of characteristics. The algorithm is written in Lagrangian coordinates for conical and cylindrical shells, on the basis of a quasilinear system of sixth order equations of a geometrically nonlinear theory of Timoshenko type [1].

The algorithm presumes the absence of strong discontinuities, i.e., of discontinuities in the first derivatives of the shell displacement, which will limit the class of admissible edge effects and permit carrying out the analysis up to the appearance of the first shock in problems where the shocks originate during wave propagation. Despite this, the proposed algorithm permits elucidation of specific properties of the wave solution in nonlinear theory. An illustrative example is given for a conical shell.

In speaking of the one-dimensional transients of conical and cylindrical shell deformation, we shall have in mind the axisymmetric processes of these objects and the plane deformations of an infinite cylindrical shell (ring).

One-dimensional transients of conical and cylindrical shell deformations have been investigated in a wave formulation in the past decade by using various models and methods within the scope of linear shell theory [2-4].

Fast-moving conical and cylindrical shell deformations have also been investigated within the scope of nonlinear shell theory [5-10, 11], but methods to analyze the specific properties of traveling waves are still almost lacking.

Indeed, mainly the application of

a) Methods of reducing the shell to a system with a finite number of degrees of freedom [12-16];

b) Finite-difference methods of integration by using a mesh determined in advance without clarification and taking account of discontinuities in the solution [17-19];

c) The method of lines [rays] with subsequent integration of a system of Runge-Kutta ordinary differential equations [20];

d) A method of approximating the tangential displacements by the linear "column" solution with the subsequent calculation of the normal displacements from the nonlinear equations [21-23], is found in existing papers which are devoted primarily to the analysis of dynamic stability problems.

These methods turned out to provide results in stability analysis, but do not allow complete characterization of the solution as a nonlinear transient of wave propagation. Hence, there is little information on how a geometrically nonlinear wave process, which is known to be small in some initial stage of the motion will differ from a linear wave process, will deviate more and more from the linear with the growth of time and become qualitatively different for high enough values of the time. Not explained are the limits of the well-founded applicability of linear theory, as well as the role of the shocks for a change in the character of the wave transient, including the appearance of large normal displacements (loss of stability).

It is shown that the mentioned questions can be clarified by using the method of characteristics within the scope of nonlinear theory. However, an algorithm is needed for this, which is adapted to the case of the presence of several pairs of families of characteristics (there are three such pairs in a nonlinear theory of Timoshenko type in the case of one-dimensional wave processes).
The algorithm proposed below consists of the following elements:

a) The representation of the initial equations [1] as a quasilinear system of first-order equations in the first derivatives of the displacement and additional formulas to calculate the displacements which are also in the coefficients of the system of equations;

b) The construction of equations governing the direction of the characteristics and the differential relations on the characteristics;

c) An algorithm of the iterative product of a computation by the second method of Masseau in the case of the presence of several pairs of families of characteristics.

Let us note that the method of characteristics has been applied in [24-26] within the scope of a linear shell theory of Timoshenko type. The directions of the characteristics are constant in linear theory and independent of the solution, but in a theory of Timoshenko type two out of the three pairs of families of characteristics coincide. During the calculation of the linear solution (for comparison with the nonlinear solution), it was noted in passing that the difference between linear solutions obtained by the method of characteristics in [25, 26] is a result of an error in one coefficient of the equations in [25].

1. Initial equations. Let us consider conical and cylindrical shells. Let h be the shell thickness, R_0 the radius of the middle surface at the shell endface, where the effect is applied, $\theta = \mathrm{const}$ is the angle between the axis and the generator of the middle surface of the conical shell ($\theta = 0$) in the case of a cylindrical shell), E is the elastic modulus, ν the Poisson's ratio, k_T a shear coefficient in a shell theory of Timoshenko type, ρ_0 the density of the shell material in the undeformed state, R_j $(j = 1, 2)$ the radii of curvature of the middle surface, t the time, τ dimensionless time. Let a coordinate system with the Lamé parameters

$$A = h = \mathrm{const}, \qquad B = R_0 + \alpha h \sin\theta$$

be selected on the shell middle surface. Hence

$$\tau = tc_2h^{-1}, \qquad c_2 = E^{1/2}\,[2\,(1+\nu)\,\rho_0]^{-1/2}$$

$$R_1 = \infty, \qquad R_2 = R_0\,(\cos\theta)^{-1} + \alpha h \,\mathrm{tg}\,\theta$$

Let us consider axisymmetric wave processes dependent on the Lagrange coordinates α and τ. Let w_1 be the dimensionless (divided by h) tangential displacement, w_2 the dimensionless (divided by h) normal displacement, and w_3 the angle of rotation of the normal.

Let us introduce the notation

$$\frac{\partial(\cdots)}{\partial\alpha} = (\cdots)', \qquad \frac{\partial(\cdots)}{\partial\tau} = (\cdots)^{\bullet}, \qquad k^2 = \frac{1}{2}(1-\nu)$$

$$\delta = hR_2^{-1}, \qquad \gamma = B'B^{-1}, \qquad \eta = \gamma w_1 + \delta w_2$$

and the new unknown terms

$$V_1 = w_1^{\bullet}, \quad V_2 = w_2^{\bullet}, \quad V_3 = w_3^{\bullet}, \quad V_4 = w_1', \quad V_5 = w_2', \quad V_6 = w_3' \qquad (1.1)$$

Utilizing the nonlinear equations [1] and the relationships

$$w_i'^{\bullet} = w_i^{\bullet\prime} \qquad (i = 1, 2, 3)$$

it is easy to derive the matrix equation

$$\mathbf{I}\mathbf{V}^{\bullet} + \mathbf{M}\mathbf{V}' - \mathbf{G} = 0 \qquad (1.2)$$

and the formula

$$w_i = \int V_{i+3}\,d\alpha + V_i\,d\tau \qquad (i = 1, 2, 3) \qquad (1.3)$$

where I is the unit matrix and $\mathbf{V}$, $\mathbf{M}$, $\mathbf{G}$ are the following matrices:

$$\mathbf{V} = \begin{Vmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{Vmatrix}, \quad \mathbf{M} = \begin{Vmatrix} 0 & 0 & 0 & -\Phi_{11} & -\Phi_{12} & -\Phi_{13} \\ 0 & 0 & 0 & -\Phi_{21} & -\Phi_{22} & 0 \\ 0 & 0 & 0 & -\Phi_{31} & 0 & -\Phi_{33} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{Vmatrix}, \quad \mathbf{G} = \begin{Vmatrix} \Phi_{10} \\ \Phi_{20} \\ \Phi_{30} \\ 0 \\ 0 \\ 0 \end{Vmatrix} \qquad (1.4)$$

The following notation has been used in (1.4)

$$\Phi_{11} = k^{-2}(1 + 3V_4 + \nu\eta), \qquad \Phi_{12} = k_T^2 w_3 + k^{-2} V_5$$
$$\Phi_{13} = {}^1/_{12} k^{-2}(2V_6 + \nu\gamma w_3) \; \Phi_{21} = \Phi_{12}, \qquad \Phi_{22} = k_T^2 + k^{-2}(V_4 + \nu\eta)$$
$$\Phi_{31} = 12\Phi_{13}, \qquad \Phi_{33} = k^{-2}(1 + 2V_4) \tag{1.5}$$
$$\Phi_{10} = k^{-2}\gamma\,[V_4 - \eta + {}^1/_2(3+\nu)(V_4^2 - \eta^2) + {}^1/_2(1-\nu)V_5^2] +$$
$$+ k^{-2}\nu\delta V_5(1 + V_4 + \eta) + k_T^2\,[(2w_3 + V_5)V_6 + \gamma(w_3 + V_5)w_3] + {}^1/_{12}k^{-2}\times$$
$$\times [(1+\nu)V_6^2 - \gamma(\gamma w_3 + \nu V_6)w_3] \tag{1.6}$$
$$\Phi_{20} = k^{-2}\gamma(1+\nu)V_4V_5 - k^{-2}\delta\,[(1 + {}^3/_2\eta)\eta + \nu(1 + {}^1/_2 V_4 + \eta)V_4 -$$
$$- {}^1/_2\nu V_5^2] + k_T^2\,[(V_6 + \gamma w_3)(1 + V_4) + \gamma V_5] - {}^1/_{12}k^{-2}\gamma\delta(\gamma w_3 + \nu V_6)w_3$$
$$\Phi_{30} = k^{-2}\gamma(1 + 2V_4 - 2\eta)V_6 - k^{-2}\gamma^{-2}(1 - \nu V_4 + \nu\eta)w_3 +$$
$$+ k^{-2}\gamma\delta\nu V_5 w_3 - 12k_T^2\,[(1 + 2V_4)w_3 + (1 + V_4)V_5]$$

The stress resultants, transverse forces, and moments can easily be expressed in terms of the dimensionless quantities introduced above. For example, in the transverse section α = const the tangential stress resultant T_{11}, the transverse force N_1 and the moment M_{11} can be represented as follows:

$$T_{11} = E_p\,[V_4 + {}^1/_2 V_4^2 + \nu(\eta + {}^1/_2\eta^2 + V_4\eta) + {}^1/_2 V_5^2 + K(w_3^2 + w_3V_5)] +$$
$$+ Dh^{-2}(V_6^2 + \nu\gamma V_6 w_3).$$
$$N_1 = E_p\,[(V_4V_5 + \nu\eta V_5) + K(w_3 + V_5 + V_4 w_3)]$$
$$M_{11} = Dh^{-1}\,[V_6 + 2V_4V_6 + \nu\gamma w_3(1 + V_4 + \eta)] \tag{1.7}$$
$$E_r = Eh(1-\nu^2)^{-1}, \qquad K = k^2k_T^2, \qquad D = Eh^3\,[12(1-\nu^2)]^{-1}$$

In substance, the matrix equation (1.2) is a quasilinear system of six first-order equations in $V_j(\alpha, \tau)$ $(j = 1, 2, \ldots, 6)$. The displacements w_j $(j = 1, 2, 3)$ are defined in terms of V_j $(j = 1, 2, \ldots 6)$ by (1.3). Six initial and three boundary conditions in the endface sections of the shell must be formulated upon application of (1.2), (1.3) in wave propagation problems. Since α is a Lagrange coordinate, then α = const in the endface sections.

The zero initial conditions

$$w_j(\alpha, 0) = 0, \qquad V_j(\alpha, 0) = 0 \qquad (j = 1, 2, 3) \tag{1.8}$$

are kept in mind below in the specific discussions. From the first group of conditions (1.8) we have

$$V_j(\alpha, 0) = 0 \qquad (j = 4, 5, 6) \tag{1.9}$$

The boundary conditions for $\alpha = \alpha_0$, α_0 = const can be formulated as follows, for example:

$$\begin{aligned} w_1(\alpha_0, \tau) &= g_1(\tau) \quad \text{or} \quad T_{11}(\alpha_0, \tau) = g_4(\tau) \\ w_2(\alpha_0, \tau) &= g_2(\tau) \quad \text{or} \quad N_1(\alpha_0, \tau) = g_5(\tau) \\ w_3(\alpha_0, \tau) &= g_3(\tau) \quad \text{or} \quad M_{11}(\alpha_0, \tau) = g_6(\tau) \end{aligned} \tag{1.10}$$

where $g_i(\tau)$ are given functions of τ. If $w_i(\alpha_0, \tau) = g_i(\tau)$ $(i = 1, 2, 3)$, then

$$V_i(\alpha_0, \tau) = g_i^{\cdot}(\tau) \qquad (i = 1, 2, 3) \tag{1.11}$$

The passage from the nonlinear to the linear formulation of the problem means neglecting all, in (1.5), and the square terms, in (1.6), dependent on V_j and

w_i ($j = 1, 2, \ldots, 6$; $i = 1, 2, 3$). Let φ_{ij} be coefficients obtained by the means mentioned from the expressions Φ_{ij}, then to go over to the application of linear theory in (1.4), the Φ_{ij} should be replaced by the following coefficients φ_{ij}:

$$\varphi_{11} = k^{-2}, \quad \varphi_{12} = \varphi_{13} = \varphi_{21} = \varphi_{31} = 0, \quad \varphi_{22} = k_T^2, \quad \varphi_{33} = k^{-2} \tag{1.12}$$

$$\varphi_{10} = k^{-2}\gamma (V_4 - \eta) + k^{-2}\delta\nu V_5, \quad \varphi_{20} = k_T^2 (V_6 + \gamma w_3 + \gamma V_5) - \\ -k^{-2}\delta(\eta + \nu V_4), \; \varphi_{30} = k^{-2}\gamma V_6 - k^{-2}\gamma^2 w_3 - 12 k_T^2 (w_3 + V_5) \tag{1.13}$$

2. Equations to realise the method of characteristics. It is known [27] that the directions of the chracteristics of a matrix equation of the form (1.2) can be determined by utilizing the eigenvalues of the equation

$$\| M - I\lambda \| = 0 \tag{2.1}$$

from which follows the equations

$$\lambda^6 - a\lambda^4 + b\lambda^2 - c = 0 \tag{2.2}$$

$$a = \Phi_{11} + \Phi_{22} + \Phi_{33}, \quad b = \Phi_{11}\Phi_{22} + \Phi_{11}\Phi_{33} + \Phi_{22}\Phi_{33} - \Phi_{12}^2 - 12\Phi_{13}^2 \\ c = \Phi_{11}\Phi_{22}\Phi_{33} - \Phi_{33}\Phi_{12}^2 - 12\Phi_{22}\Phi_{13}^2 \tag{2.3}$$

The characteristics in the α,τ-plane are the curves $d\alpha / d\tau = \lambda_j$ ($j = 1, 2, \ldots, 6$). Here λ_j are the roots of (2.2). The condition of hyperbolicity of the initial system (1.2) is identical to the condition of all roots in (2.2) being real, and can be represented as

$$({}^1/_2 q)^2 - ({}^1/_3 p)^3 \leqslant 0 \tag{2.4}$$

$$p = {}^1/_3 a^2 - b, \qquad q = {}^2/_{27} a^3 - {}^1/_3 ab + c \tag{2.5}$$

The roots of (2.2) can be calculated from the formula

$$\lambda_j = \pm \{{}^1/_3 a + 2({}^1/_3 p)^{1/2} \cos [{}^1/_3 (\varphi + 2n\pi)]\}^{1/2} \qquad (n = 1, 2, 3) \tag{2.6}$$

$$\varphi = \arccos [{}^1/_2 q ({}^1/_3 p)^{-3/2}] \tag{2.7}$$

Let the roots of (2.6) be enumerated as follows:

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_6 \tag{2.8}$$

The functions Φ_{ij} depend on the desired solution, hence the λ_j also depend on the desired solution.

The matrix $M - I\lambda$ is singular, hence, there exists the eigenvector [27]

$$C^{(i)} = \{C_1^{(i)}, C_2^{(i)}, C_3^{(i)}, C_4^{(i)}, C_5^{(i)}, C_6^{(i)}\}$$

for which

$$C^{(i)} M = \lambda_i C^{(i)} \qquad (i = 1, 2, \ldots, 6) \tag{2.9}$$

Equation (2.9) decomposes into the systems

$$\begin{Vmatrix} \lambda_j^2 - \Phi_{11} & -\Phi_{21} & -\Phi_{31} \\ -\Phi_{12} & \lambda_j^2 - \Phi_{22} & 0 \\ -\Phi_{13} & 0 & \lambda_j^2 - \Phi_{33} \end{Vmatrix} \begin{Vmatrix} C_1^{(j)} \\ C_2^{(j)} \\ C_3^{(j)} \end{Vmatrix} = 0 \tag{2.10}$$

$$C_k^{(7-j)} = C_k^{(j)} \qquad (k = 1, 2, 3) \tag{2.11}$$

$$C_4^{(i)} = -\lambda_i C_1^{(i)}, \quad C_5^{(i)} = -\lambda_i C_2^{(i)}, \quad C_6^{(i)} = -\lambda_i C_3^{(i)} \qquad (i = 1, 2, \ldots, 6) \tag{2.12}$$

The matrix equation (2.10) has the solutions

$$C_k^{(j)} = (-1)^{n-k} \frac{N_k}{N_n} C_n^{(j)} \qquad (j = 1, 2, 3;\ k = 1, 2, 3) \tag{2.13}$$

where N_k, N_n are minors of the matrix of the system (2.10) obtained by deleting its k-th and n-the column, respectively. Let us give the components $C_1^{(1)} = 1$, $C_3^{(2)} = 1$, $C_2^{(3)} = 1$, and let us evaluate the remaining components with the subscripts $k = 1, 2, 3$ by (2.13). We have

$$
\begin{aligned}
&C_2^{(1)} = \Phi_{12} (\lambda^2 - \Phi_{22})^{-1}, \qquad C_3^{(1)} = \Phi_{13} (\lambda_1^2 - \Phi_{33})^{-1} \\
&C_1^{(2)} = 12\Phi_{13} (\lambda_2^2 - \Phi_{22}) [(\lambda_2^2 - \Phi_{11}) (\lambda_2^2 - \Phi_{22}) - \Phi_{12}^2]^{-1} \\
&C_2^{(2)} = 12\Phi_{12}\Phi_{13} [(\lambda_2^2 - \Phi_{11}) (\lambda_2^2 - \Phi_{22}) - \Phi_{12}^2]^{-1} \\
&C_1^{(3)} = \Phi_{12} (\lambda_3^2 - \Phi_{33}) [(\lambda_3^2 - \Phi_{11}) (\lambda_3^2 - \Phi_{33}) - 12\Phi_{13}^2]^{-1} \\
&C_3^{(3)} = \Phi_{12}\Phi_{13} [(\lambda_3^2 - \Phi_{11}) (\lambda_3^2 - \Phi_{33}) - 12\Phi_{13}^2]^{-1}
\end{aligned}
\qquad (2.14)
$$

All the remaining components are evaluated by means of the relationships (2.12). Multiplying (1.2) on the left by the eigenvector $C^{(i)}$, we obtain

$$\mathbf{C}^{(i)}\mathbf{V}^{\cdot} + \mathbf{C}^{(i)}\mathbf{M}\mathbf{V}' - \mathbf{C}^{(i)}\mathbf{G} = 0$$

Taking account of (2.9), we have

$$\mathbf{C}^{(i)}(D^i\mathbf{V}^i) - \mathbf{C}^{(i)}\mathbf{G} = 0 \qquad (2.15)$$

$$D^i\mathbf{V}^i = \mathbf{V}^{\cdot} + \lambda_i\mathbf{V}' \qquad (2.16)$$

Equation (2.15) determines the differential relations on the characteristics.

We obtain the characteristic directions to the accuracy of linear theory from (2.1) after having replaced the coefficients Φ_{ij} from (1.5) in the matrix $\mathbf{M}$ by the coefficients φ_{ij} (1.12)($i, j = 1, 2, 3$). These directions are constant

$$\lambda_1 = \lambda_2 = k^{-1}, \quad \lambda_3 = k_T, \quad \lambda_4 = -k_T, \quad \lambda_5 = \lambda_6 = -k^{-1} \qquad (2.17)$$

This means that multiple roots appear in the linear formulation. Let us determine the eigenvectors from (2.10)-(2.12), where the Φ_{ij} in (1.5) have been replaced by the φ_{ij} in (1.12) and from the condition that linearly independent eigenvectors would correspond to the multiple roots. Such solutions of (2.10) are

$$C_1^{(1)} = 1, \quad C_2^{(1)} = C_3^{(1)} = 0, \quad C_1^{(2)} = C_2^{(2)} = 0, \quad C_3^{(2)} = 1, \quad C_1^{(3)} = C_3^{(3)} = 0$$
$$C_2^{(3)} = 1 \qquad (2.18)$$

3. Algorithm of the numerical realization of the method of characteristics. Following Courant [28], let us utilize a matrix description of the algorithm. The unit vector $\mathbf{s}^{(i)}$ $(i = 1, 2, \ldots, 6)$ in the direction of the i-th characteristic has the following projections in the $\alpha\tau$-plane:

$$\lambda_i (1 + \lambda_i^2)^{-1/2} \text{ на } \alpha\text{-ось}, \qquad (1 + \lambda_i^2)^{-1/2} \text{ на } \tau\text{-ось} \qquad (3.1)$$

The derivative along the i-th characteristic, i.e., in the direction $\mathbf{s}^{(i)}$ has the following form:

$$\mathbf{s}^{(i)} \cdot \nabla\mathbf{V} = \lambda_i (1 + \lambda_i^2)^{-1/2}\mathbf{V}' + (1 + \lambda_i^2)^{-1/2}\mathbf{V}^{\cdot} \qquad (3.2)$$

Utilizing the notation (2.16), we obtain

$$\mathbf{s}^{(i)} \cdot \nabla\mathbf{V} = (1 + \lambda_i^2)^{-1/2}(D^i\mathbf{V}^i) \qquad (3.3)$$

Let us construct the algorithm of a finite-difference integration of the matrix equation (2.15), and let us evaluate w_k by means of (1.3). Let us utilize the so-called standard technique for which the main mesh in the $\alpha\tau$-plane is selected along the characteristics having the directions λ_1, λ_6. Let us introduce the numbering of the nodes in this mesh indicated in Fig. 1. Let us consider the triple of nodes $P_{m,n}$, $P_{m-1,n}$, $P_{m,n-1}$. Let the coordinates of the nodes $P_{m-1,n}$, $P_{m,n-1}$ and the quantities V_j $(j = 1, 2, \ldots, 6)$, w_k $(k = 1, 2, 3)$ at these nodes be known, and the coordinates of the node $P_{m,n}$ and the quantities V_j $(j = 1, 2, \ldots, 6)$, w_k $(k = 1, 2, 3)$ at this node unknown and subject to calculation. For brevity in writing the computational algorithm

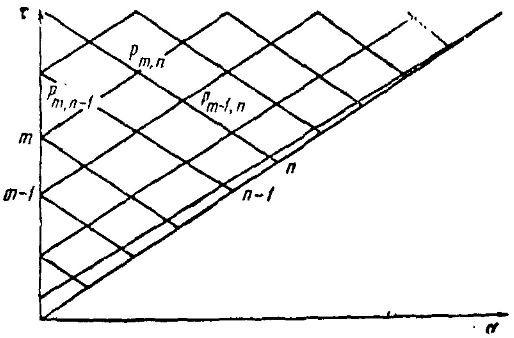

Fig. 1.

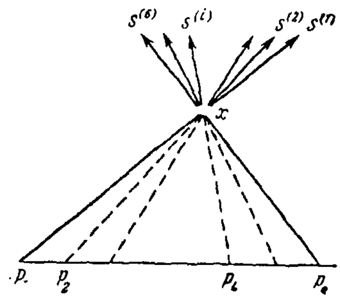

Fig. 2.

let us introduce the following notation for the nodes (Fig. 2)

$$P_{m,n} \equiv X, \qquad P_{m,n-1} \equiv p_1, \qquad P_{m-1,n} \equiv p_6$$

Let us still introduce the auxiliary points p_2, p_3, p_4, p_5 as is indicated in Fig. 2. In the finite-difference approximation, the derivative along the i-th characteristic has the form

$$s^{(i)} \cdot \nabla V(p_i^*) = [V(X) - V(p_i)]\,[(1+\lambda_i^2)^{1/2}\Delta\tau_i]^{-1} \qquad (i = 1, 2, \ldots, 6) \tag{3.4}$$

$$\Delta\tau_i = \tau(X) - \tau(p_i) \tag{3.5}$$

and ∇V (p_i^*) should be understood to be the middle derivative along the i-th characteristic on the segment between the nodes X and p_i. Equating (3.3) and (3.4) we obtain

$$D^i V^i(p_i^*) = [V(X) - V(p_i)]\,(\Delta\tau_i)^{-1} \tag{3.6}$$

Substituting (3.6) into (2.15), we have the following finite-difference equation in the matrix form

$$C^{(i)}(p_i^*)\,V(X) = C^{(i)}(p_i^*)\,[V(p_i) + G(p_i^*)\Delta\tau_i] \tag{3.7}$$

Here and henceforth, $G(p_i^*)$, $\lambda(p_i^*)$, $C^{(i)}(p_i^*)$ should be understood to be quantities calculated by means of (1.4), (2.6) and (2.13), respectively, in which (in conformity with the second method of Masseau [29]) the following approximate representations have been introduced

$$V_k = V_k(p_i^*) = 1/2\,[V_k(X) + V_k(p_i)], \qquad w_j = w_j(p_i^*) = 1/2\,[w_j(X) + w_j(p_i)]$$
$$\alpha = \alpha(p_i^*) = 1/2\,[\alpha(X) + \alpha(p_i)] \tag{3.8}$$

Let us now eliminate values of V at the auxiliary points p_2, p_3, p_4, p_5 from the system (3.7). From geometric considerations it follows that

$$\tau(X) = [\alpha(p_1) - \alpha(p_6) - \lambda_1(p_1^*)\tau(p_1) + \lambda_6(p_6^*)\tau(p_6)]\,[\lambda_6(p_6^*) - \lambda_1(p_1^*)]$$
$$\alpha(X) = \alpha(p_6) - [\tau(X) - \tau(p_6)]\,\lambda_6(p_6^*) \tag{3.9}$$
$$\alpha(p_i) = \{\alpha(X) + [\tau(p_6) - \tau(X) - (\tau(p_6) - \tau(p_1))\,(\alpha(p_6) -$$

$$-\alpha(p_1))^{-1}\alpha(p_6)]\lambda_i(p_i^*)\}[1-(\tau(p_6)-\tau(p_1))-\alpha(p_1))^{-1}\lambda_1(p_1^*)]^{-1}$$

$$\tau(p_i)=[\alpha(p_i)-\alpha(X)][\lambda_i(p_i^*)]^{-1}+\tau(X) \qquad (3.10)$$

We find the values of the desired functions at the point p_i by interpolation between the points p_1 and p_6, which results in thefollowing formulas:

$$V_k(p_i)=(1-Q_i)V_k(p_6)+Q_iV_k(p_1), \quad w_k(p_i)=(1-Q_i)w_k(p_6)+ + Q_iw_k(p_1) \qquad (3.11)$$

$$Q_i=[\alpha(p_6)-\alpha(p_i)][\alpha(p_6)-\alpha(p_1)]^{-1} \qquad (3.12)$$

Substituting (3.10) into (3.6), we obtain

$$\mathbf{C}^{(i)}(p_i^*)\mathbf{V}(X)=\mathbf{C}^{(i)}(p_i^*)[(1-Q_i)\mathbf{V}(p_6)+Q_i\mathbf{V}(p_1)+\mathbf{G}(p_i^*)\Delta\tau_i] \qquad (3.13)$$
$$(i=1,2,3,\ldots,6)$$

We take the integrals (1.3) along the characteristics with the directions λ_1 and λ_6 by means of the formula

$$w_k(X)={}^1/_2\{[V_{k+3}(p_1^*)\lambda_1(p_1^*)+V_k(p_1^*)]\Delta\tau_1+w_k(p_1)+ + [V_{k+3}(p_6^*)\lambda_6(p_6^*)+V_k(p_6^*)]\Delta\tau_6+w_k(p_6)\} \qquad (k=1,2,3) \qquad (3.14)$$

The problem now is to produce an iterative algorithm to find such coordinates α, τ of the node X, as well as the quantities $V_j(X)$, $j=1,2,\ldots 6$ and $w_k(X)$, $k=1,2,3$, for which the equalities (3.13), (3.14) are satisfied with a numerically assigned accuracy.

Let us consider the calculation of the zero approximation, utilizing the approximation formulas

$$\lambda_k(p_i^*)=\lambda_k(p_1), \quad w_j(p_i^*)=w_j(p_1), \quad V_k(p_i^*)=V_k(p_1) \qquad (i=1,2,3) \qquad (3.15)$$
$$\lambda_k(p_i^*)=\lambda_k(p_6), \quad w_j(p_i^*)=w_j(p_6), \quad V_k(p_i^*)=V_k(p_6) \qquad (i=4,5,6)$$

in the following way.

First, let us calculate the coordinates of the node X in a zero approximation by means of (3.9), in whose right sides the $\lambda_k(p_i^*)$ are approximated in the form (3.15). Then let us evaluate $w_j(X)$ by (3.14) utilizing the approximation (3.15) in the right side. The $\Delta\tau_i$ are hence calculated by means of (3.5). Applying the approximation (3.15), we find $\mathbf{C}^{(i)}(p_i^*)$ in a zero approximation. Then we evaluate $\mathbf{G}(p_i^*)$ by utilizing the $\alpha(X)$, $w_j(X)$ in the zero approximation already found and by approximating $V_k(p_i^*)$ by (3.15). It now turns out to be possible to calculate $\mathbf{V}(X)$ in a zero approximation form (3.12).

The improved coordinates of the node X, the improved vectors $\mathbf{C}^{(i)}(p_i^*)$, $\mathbf{G}(p_i^*)$, $\mathbf{V}(X)$ are calculated alternately in the next approximations. Formulas (3.8) and all the necessary quantities are used to calculate each quantity, in the approximation in which they are available at this stage of the iteration process. The iteration is continued until the difference in the last and next-to-last approximations of the desired quantities becomes less than some given value.

Let us examine the question of imposing the boundary conditions at the shell edge $\alpha=\alpha_0$, $\alpha_0=\text{const}$. The boundary $\alpha=\alpha_0$ is a time-like line in the $\alpha\tau$-plane, and hence, only three of the six vectors $\mathbf{s}^{(i)}$ are directed to one side. Therefore, three equations with six unknown V_i follow from the matrix equation (3.13) on the boundary $\alpha=\alpha_0$. We add the three boundary conditions (1.10) to these three equations, whereupon we have a system of six equations in six components of the vector $\mathbf{V}$. Let us construct the following vectors

$$G_i^*=\delta_{ij}V_i, \qquad V_i^*=(1-\delta_{ij})V_i \qquad (i=1,2,3,\ldots,6)$$

where δ_{ij} is the Kronecker symbol, and j is the subscript taking on the three values which correspond to the three components V_j, given on the boundary, or expressed on the boundary (by using the boundary conditions) in terms of given functions and the remaining components of the vector $\mathbf{V}$. In other words, $\mathbf{V}^*$ is an unknown three-component vector, and $\mathbf{G}^*$ is a three-component vector whose components are either known, or expressed in terms of the components V_i.

Representing the six-component vector $\mathbf{V}$ as

$$\mathbf{V} = \mathbf{V}^* + \mathbf{G}^* \tag{3.16}$$

we obtain the following equation from (3.13):

$$\mathbf{C}^{(i)}(p_i^*)\,\mathbf{V}^*(X) = \mathbf{C}^{(i)}(p_i^*)\,[(1 - Q_i)\,\mathbf{V}(p_6) + Q_i\mathbf{V}(p_1) + \\ + \mathbf{G}(p_i^*)\,\Delta\tau_i - \mathbf{G}^*(X)] \tag{3.17}$$

where $X \equiv P_{m,m}$, $P_1 \equiv P_{m-1,m-1}$ in the $\tau \geqslant 0, \alpha \geqslant \alpha_0$ domain denote the boundary points, $P_6 \equiv P_{m-1,m}$, $i = 4, 5, 6$ and $X \equiv P_{m,m}$, $P_6 \equiv P_{m-1,m-1}$ denote the boundary points, $P_1 = P_{m,m-1}$, $i = 1, 2, 3$. If some of the three dimensionless displacements are not given on the boundary $\alpha = \alpha_0$ then these unassigned dimensionless displacements are calculated by (3.14), which becomes on the boundary

$$w_k(X) = [V_{k+3}(p_i^*)\,\lambda_i(p_i^*) + V_k(p_i^*)]\,\Delta\tau_i + w_k(p_i) \tag{3.18}$$

For $\tau \geqslant 0$, $\alpha \geqslant \alpha_0$ in (3.18) one should take $i = 6$ and $i = 1$ for $\tau \geqslant 0, \alpha \leqslant \alpha_0$ The interpolation process on the boundary can be accomplished just as for the interior points described above.

4. Numerical example. Let us consider the wave transient for $\tau \geqslant 0$ in the semi-infinite ($\alpha \geqslant 0$) conical shell. Let us give the zero initial conditions (1.8) and the following functions in the boundary conditions (1.10):

$$g_2(\tau) = g_3(\tau) = 0, \qquad g_4(\tau) = 2T\pi^{-1}\operatorname{arctg}(m\tau) \tag{4.1}$$

Let us represent the vector $\mathbf{V}$ on the boundary in the form (3.16), where in this case

$$\mathbf{V}^*(0,\tau) = \begin{Vmatrix} V_1(0,\tau) \\ 0 \\ 0 \\ 0 \\ V_5(0,\tau) \\ V_6(0,\tau) \end{Vmatrix} \qquad \mathbf{G}^*(0,\tau) = \begin{Vmatrix} 0 \\ V_2(0,\tau) \\ V_3(0,\tau) \\ V_4(0,\tau) \\ 0 \\ 0 \end{Vmatrix} \tag{4.2}$$

From (1.11), (1.7), (4.1) follows

$$V_2(0,\tau) = V_3(0,\tau) = 0 \tag{4.3}$$

$$V_4(0,\tau) = E_p^{-1} g_4(\tau) - {}^3/_2 V_4^2(0,\tau) - \nu\,[\eta(0,\tau) + {}^1/_2\eta^2(0,\tau) + \\ + V_4(0,\tau)\,\eta(0,\tau)] - {}^1/_2 V_5^2(0,\tau) - {}^1/_{12} V_6^2(0,\tau) \tag{4.4}$$

In the linear formulation of the problem, (4.4) simplifies to the following:

$$V_4(0,\tau) = E_p^{-1} g_4(\tau) - \nu\eta(0,\tau) \tag{4.5}$$

We present the results of a numerical computation obtained for a conical shell with the following values for the coefficients:

$$\theta = -30^\circ, \quad h/R_0 = {}^1/_{50}, \quad k_T^2 = 0.87, \quad \nu = {}^1/_3, \quad T/Eh = 0.01, \quad m = 5$$

Partitioning of the nodes with the spacing $\Delta\tau_1 = 0.1$ was applied along the front. The coordinates of the nodal points $P_{m,n}$ (Fig. 1) and the desired quantities at these points were found at a series of points $m = 1, 2, 3, \ldots$ along the characteristic with the direction λ_1 on each of which the computation was carried out alternately at the points $n = 1, 2, 3, \ldots$

The computations were performed in parallel in a nonlinear and linear formulation, where the partitioning of the nodes on the first front $\tau = k^{-2}\alpha$ was identical. The results of the computation show that by nonlinear theory V_1 and V_4 along the chracteristics with the direction λ_1 differ slightly in the initial stage from the V_1 and V_4 from linear theory. However, the characteristics of identical order number m by linear and nonlinear theory do not coincide in the α, τ-plane.

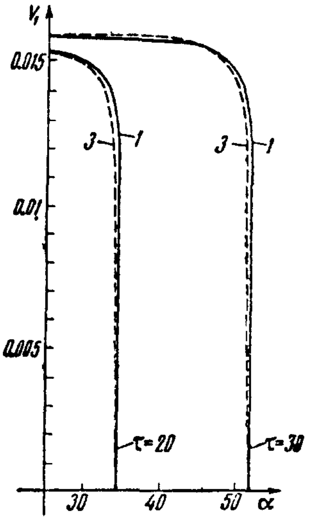

Fig. 3.

The quantitative difference between the linear and nonlinear solution grows as time increases in the initial stage, mainly for this reason. The qualitative difference between the linear and nonlinear solution occurs at the intersection of the characteristics at $\tau = 30$ (Fig. 1), which results in the appearance of a discontinuity in the components V_1, V_4 of the nonlinear solution. In other words, a shock originates in this example for $\tau = 30$. However, quantitative results by the nonlinear and linear theories differ noticeably in the initial stage of the motion only in a narrow band behind the first front (Fig. 3). The solid lines in Fig. 3 picture the linear theory results, and the dashed lines are linear theory for $\tau = 20$, $\tau = 30$. Let us note that the shock occurs at the first front always upon the application of a tensile type tangential effect, and never occurs upon application of a compressive type tangential effect.

BIBLIOGRAPHY

1. Ainola, L. Ia., Variational methods for nonlinear equations of motion of shells. PMM Vol. 32, No. 1, 1968.
2. Ainola, L. and Nigul, U. Wave processes of elastic plate and shell deformation. Izv. Akad. Nauk, ESSR, Ser. Fiz.-matem. i Tekh. Nauk, 1965.
3. Alumiae, N. A., Deformation transients of elastic shells and plates. Trans. 6-th All-Union Conf. on Shell and Plate Theory, Baku, 1966. Nauka, Moscow, 1966 pp. 883-889.
4. Nigul, U. K., Wave deformation processes in shells and plates. Trans. 7th All-Union Conf. on Shell and Plate Theory. Dnepropetrovsk, 1969. "Nauka", 1970.
5. Vol'mir, A. S., Problems of the theory of shells and plates stability. Trans. 6-th All-Union Conf. on Plate and Shell Theory. Baku, 1966. "Nauka", 1966.
6. Vol'mir, A. S., Nonlinear problems of the theory of deformation transients of shells and the stability problem. In: Deformation Transients of Shells and Plates. (pp. 51-66) Tallin, 1967.
7. Stein, M., Some recent advances in the investigation of shell buckling. J. AIAA Vol. 6, No. 12, 1968.

8. Vol'mir, A. S., Plate and Shell Stability. In: Structural Mechanics in the USSR. 1917-1967, Moscow, Stroiizdat, (pp. 259-279) 1969.

9. Agamirov, V. L., Survey of research on the stability of structures under pulse loading. In: Analysis of Spatial Structures, No. 12, Moscow, Stroiizdat, (pp. 186-200), 1969.

10. Vol'mir, A. S., Modern problems of shell stability and dynamics. Stroit. Mekh. i Raschet Sooruzh., No. 2, 1970.

11. Vol'mir, A. S., Stability of Deformable Systems. 2d. Ed., Moscow, "Nauka" 1967.

12. Vol'mir, A. S., On the stability of cylindrical shells under dynamic loading. Dokl. Akad. Nauk. SSSR. Vol. 123, No. 5, 1958.

13. Borisenko, V. I., On the stability of a cylindrical shell under longitudinal impact. Prikl. Mekh., Vol. 1, No. 5, 1965.

14. Protsenko, O. P. and Pavlovskii, V. S., Dynamic stability of a closed cylindrical shell under combined periodic loading. In: Stability Problems in Structural Mechanics. Moscow, Stroiisdat, pp. 296-305, 1965.

15. Grigoliuk, E. I. and Srebovskii, A. I., Thin circular cylindrical shells subjected to an external pressure pulse. Inzh. Zh., Mekh. Tverd. Tela, №3, 1968.

16. Kornev, V. M., On the buckling modes of elastic shells under intensive loading. Izv. Akad. Nauk. SSSR, Mekh. Tverd. Tela, No. 2, 1969.

17. Vol'mir, A. S. and Gorshtein, M. S., Behavior of elastic cylindrical shells subjected to a plane acoustic wave. Inzh. Zh. Vol. 5, No. 6, 1965.

18. Vol'mir, E. A., Behavior of a circular cylindrical panel subjected to a shock wave in a fluid. Izv. Akad. Nauk. SSSR, Mekh. Tverd. Tela, No. 1, 1969.

19. Gorshkov, A. G. and Grigoliuk, E. I., Khromushkin, A. V. and Shkliarchuk, F. N., Motion of a cylindrical panel due to an acoustic shock wave. Izv. Akad. Nauk. SSSR., Mekh. Tverd. Tela, No. 4, 1969.

20. Valishvili, N. V. and Silkin, V. B., Application of the method of lines [rays] to solve nonlinear problems of shallow shell dynamics. Izv. Akad. Nauk. SSSR, Mekh. Tverd. Tela, No. 3, 1970.

21. Agamirov, V. L., and Vol'mir, A. S., On the stability of a cylindrical shell under longitudinal impact. Dokl. Akad. Nauk. SSSR, Vol. 157, No. 2, 1964.

22. Agamirov, V. L. and Vol'mir A. S., Behavior of cylindrical shells under longitudinal impact. In: Stability Problems in Structural Mechanics, pp. 143-152, Stroiizdat, 1965.

23. Kornev, V. M., On the buckling modes of an elastic column under impact. PMTF, No. 3, 1968.

24. Leonhard, R. W. and Budiansky, B., On traveling waves in beams. NACA Report 1173, 1954.

25. Spillers, W. R., Wave propagation in a thin cylindrical shell. Trans. ASME, Ser. E, J. Appl. Mech., Vol. 32, No. 2, 1965.

26. Chou, P. C., Analysis of axisymmetrical motions of cylindrical shells by the method of characteristics. J. AIAA, Vol. 6, No. 8, 1968.

27. Courant, R., Partial Differential Equations. Moscow, "Mir, " 1964.

28. Courant, R. Isaacson, E. and Rees, M., On the solution of nonlinear hyperbolic differential equations by finite differences. Comm. Pure and Applied Math., Vol5, No. 3, 1952.

29. Berzin, I. S. and Zhidkov, N. P. Calculation Methods, Moscow, Fizmatgiz, Vol. 2, 1960.

Translated by M. D. F.

SPECTRUM OF THE SYSTEM DESCRIBING OSCILLATIONS OF A SHELL OF REVOLUTION

PMM Vol. 35, No. 4, 1971, pp. 701-717
A. G. ASLANIAN and V. B. LIDSKII
(Moscow)
(Received November 13, 1970)

The relationship between spectra of moment and momentless [membrane] systems of differential equations which describe the characteristic oscillations of shells of revolution is examined.

For the eigenvalues of the lower series the oscillation theorem is proven. Conditions are found for which the lower series of frequencies of the momentless system has a finite limit point.

A number of papers are devoted to finding the frequencies of characteristic oscillations of a thin shell by the small parameter method (see Bibliography).

In this paper some mathematical problems are examined which are connected with the problem of finding the characteristic frequencies for a shell of revolution. In this case the characteristic oscillations with m waves along the parallel are described by the following system of equations [1, 5]:

$$\begin{gathered}
-u'' - \frac{B'}{B}u' - \frac{m(1+\sigma)}{2B}v' - \left[\left(\frac{B'}{B}\right)' + (1-\sigma)\left(\frac{1}{R_1R_2} - \frac{m^2}{2B^2}\right)\right]u + \\
+ \frac{mB'}{B^2}\frac{3-\sigma}{2}v + \left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right)w' + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)' w = \lambda u \\
-\frac{1-\sigma}{2}v'' + \frac{m}{B}\frac{(1+\sigma)}{2}u' - \frac{1-\sigma}{2}\frac{B'}{B}v' + \frac{mB'}{B^2}\frac{3-\sigma}{2}u - \\
-\left[\frac{1-\sigma}{2}\left(\frac{B'}{B}\right)' + \frac{1-\sigma}{R_1R_2} - \frac{m^2}{B^2}\right]v - \frac{m}{B}\left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right)w = \lambda v \\
\mu^4\frac{1}{B}\left(\frac{d}{ds}B\frac{d}{ds} - \frac{m^2}{B}\right)\frac{1}{B}\left(\frac{d}{ds}B\frac{d}{ds} - \frac{m^2}{B}\right)w - \left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right)\frac{du}{ds} - \\
-\left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right)\frac{B'}{B}u - \frac{m}{B}\left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right)v + \left(\frac{1}{R_1^2} + \frac{2\sigma}{R_1R_2} + \frac{1}{R_2^2}\right)w = \lambda w
\end{gathered} \tag{0.1}$$

Here u, v, w are the projections of the displacement of a point on the directions of the meridian, the parallel and the normal to the shell, respectively; s is the length of the meridian arc, $a \leqslant s \leqslant b$; $B(s)$ is the distance from the meridian to the axis of revolution; $R_1(s)$ and $R_2(s)$ are the principal radii of curvature of the shell

$$\frac{1}{R_1} = -\frac{B''}{\sqrt{1-B'^2}}, \qquad \frac{1}{R_2} = \frac{\sqrt{1-B'^2}}{B}, \qquad \lambda = (1-\sigma^2)\frac{\gamma}{E}p^2, \quad \mu^4 = \frac{h^2}{12} \tag{0.2}$$

where E is Young's modulus, σ is Poisson's ratio, γ is the density, p is the frequency of oscillations, h is the thickness of the shell and μ is the small parameter. The system